

Independent Set

[Kleinberg and Tardos 2005]

Input: Social Networks $(\mathcal{V}, \mathcal{E})$

Chapter 10.2 and 10.4

Output: $\mathcal{V}' \subseteq \mathcal{V}$

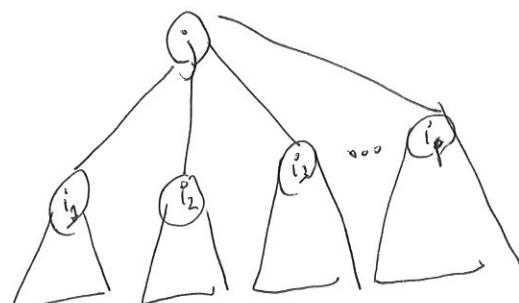
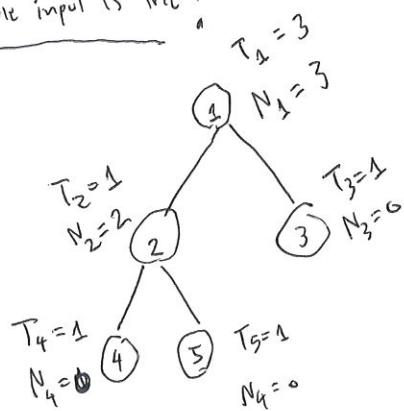
Constraint: For any $u, v \in \mathcal{V}', \{u, v\} \notin \mathcal{E}$ [We don't want a friend to be in the party]

Objective Function: Maximize $|\mathcal{V}'|$ [We want to maximize # persons who join the party]

NP-Hard = Unlikely to be solved.

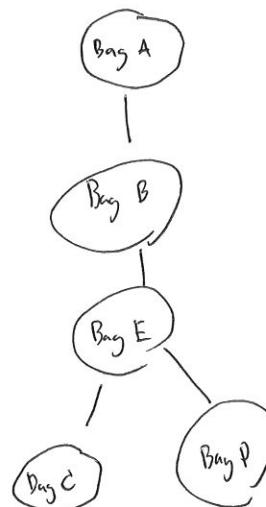
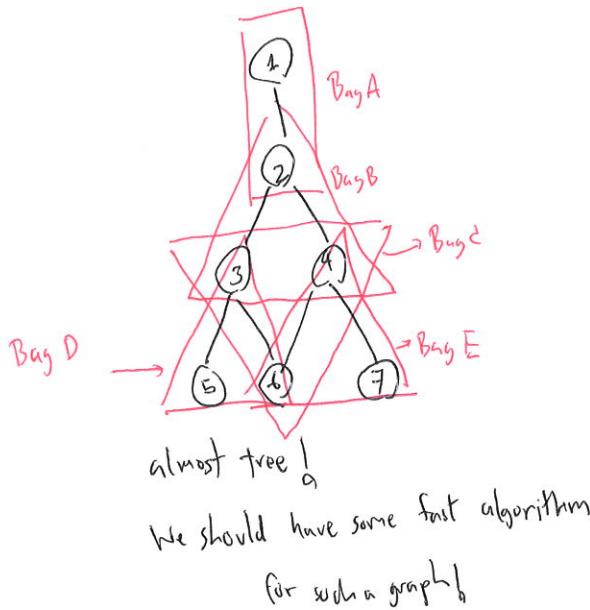
When the input is tree?

Ex



Hard problem is easy to solve when the input graph is a tree.

How about when the input graph is close to a tree?



a tree of bags
solving problems here!

Tree Decomposition

Input: Social Network (\bar{V}, E)

Output: Tree (β, Σ)

↑
set of bags ↑
links between bags

For each bag $\beta_i \in \beta$, a set of nodes associating to the bag $\bar{V}_i \in \bar{V}$

Ex $\bar{V} = \{1, 2, 3, 4, 5, 6, 7\}$

$$E = \{\{1, 2, 3\}, \{2, 3\}, \{2, 4\}, \{3, 5\}, \{3, 6\}, \{4, 6\}, \{4, 7\}\}$$



$$\beta = \{\beta_1, \dots, \beta_5\} \quad \Sigma = \{\{\beta_1, \beta_2\}, \{\beta_2, \beta_5\}, \{\beta_3, \beta_5\}, \{\beta_4, \beta_5\}\}$$

$$\bar{V}_1 = \{1, 2\}, \bar{V}_2 = \{2, 3, 4\}, \bar{V}_3 = \{3, 5, 6\}, \bar{V}_4 = \{4, 6, 7\}, \bar{V}_5 = \{3, 4, 6\}.$$

Constraint

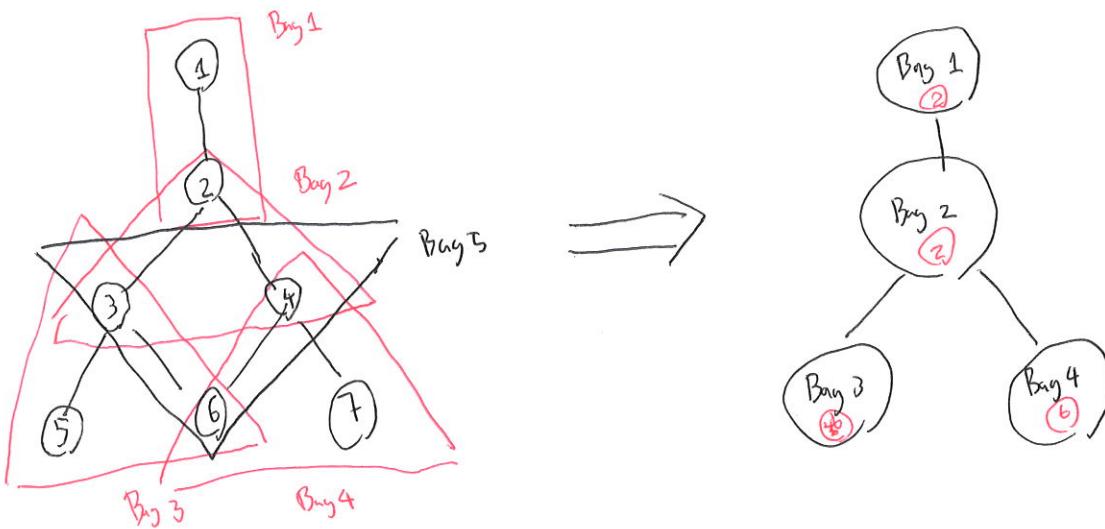
1. Node Coverage (all nodes must be in some bags)

$$\bigcup_i \bar{V}_i = \bar{V}$$

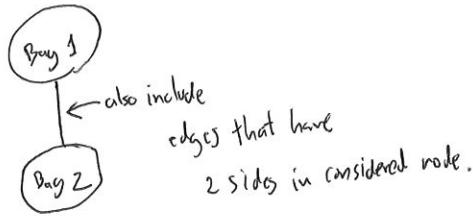
2. Edge Coverage (all edges must be in some bags)

For all $\{u, v\} \in E$, there must be some bags β_i such that $u, v \in \bar{V}_i$

3. Coherence (Bags that have a specific node must connect to each other.)



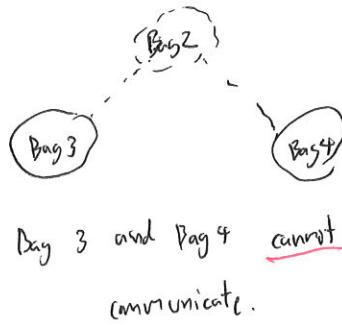
Graph induced by bags contain ②



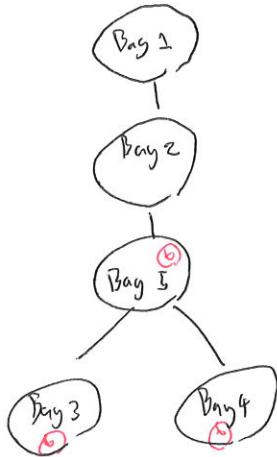
Bag 1 and Bag 2 can communicate each other

This is not a tree decomposition!

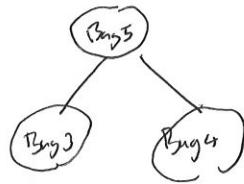
Graph induced by bags contain ⑥



With Bag 5!



Graph induced by bags contain ⑥



Bag 3, 4, 5 can connect together by 2 edges.

Formal Definition

Assume that node i contains in $V_{i_1}, V_{i_2}, \dots, V_{i_q}$

$$\beta^{(i)} := \{ \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_q} \}$$

$$\Sigma^{(i)} := \{ \{\beta_i, \beta_j\} \in \Sigma : \beta_i, \beta_j \in \beta^{(i)} \}$$

Induced subgraph by node i : $(\beta^{(i)}, \Sigma^{(i)})$

Cohesive: All induced subgraphs by node i are connected.

Objective Function

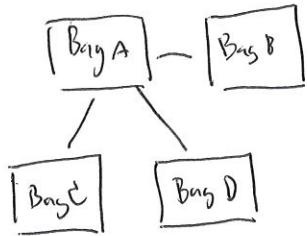
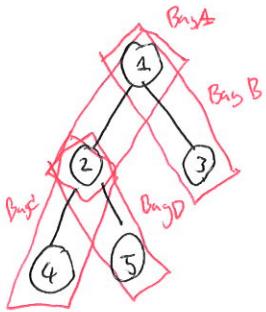
Minimize the size of largest bag

$$\text{Tree width} = \max_i |\Sigma_i| - 1$$

Minimize Tree width.

Ex Treewidth of our example = $3-1=2$

Treewidth of a tree = $2-1=1$

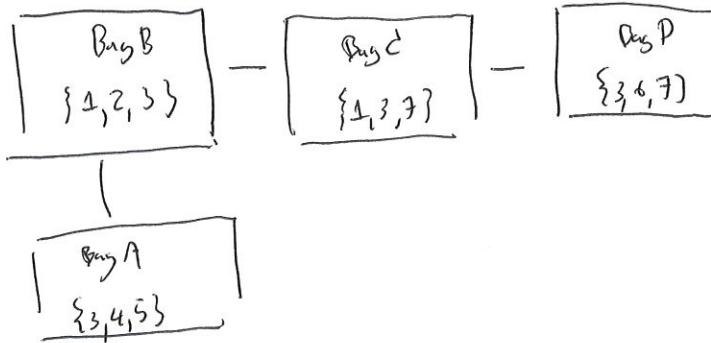
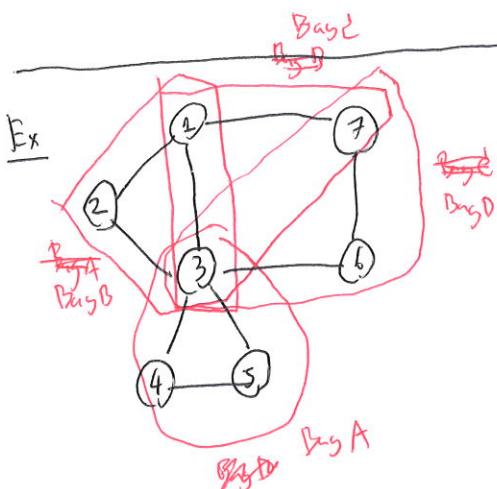


Treewidth = 1 \rightarrow tree

Treewidth = 2 \rightarrow close to tree

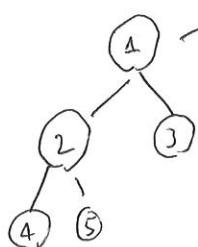
\vdots

large treewidth \longrightarrow not close to tree



Treewidth = $3-1=2$

Back to independent set



To calculate T_1 and N_1 , we care if (2) and (3) are taken. We don't care if (4) or (5) are taken.

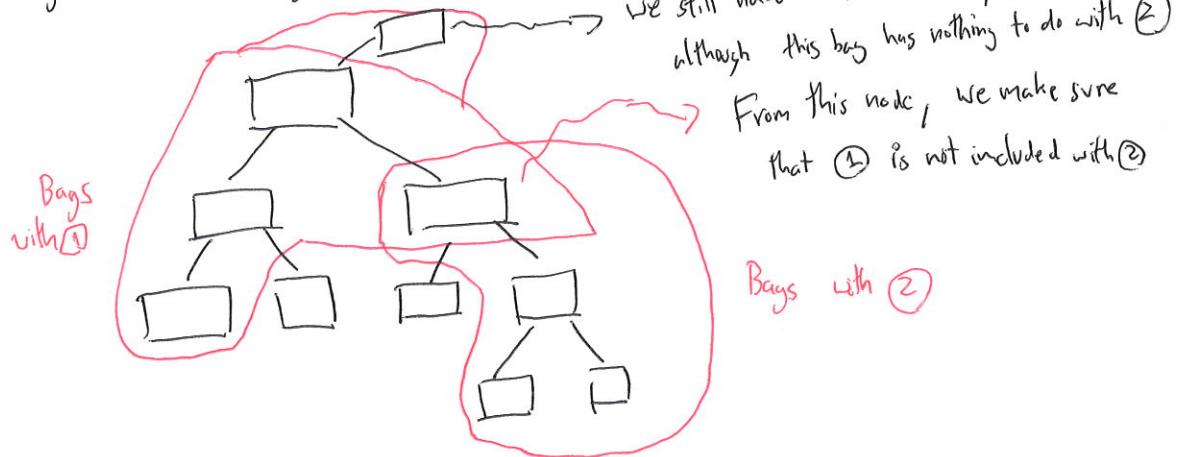


We don't have direct communication from (4) or (5) to (1)

① is adjacent to ② $\rightarrow \{1, 2, 3\} \in E$

① and ② must be together in some bags (edge coverage)

Tree of bags



1	2	3	Max # nodes
x	x	x	2
x	x	✓	2
x	✓	x	3
x	✓	✓	N/A
✓	x	x	3
✓	x	✓	N/A
✓	✓	x	N/A
✓	✓	✓	N/A

1	3	7	Max # nodes
x	x	x	1
x	x	✓	1
x	✓	x	2
✓	x	x	2
✓	x	✓	N/A
✓	✓	x	N/A
✓	✓	✓	N/A

3	Max # Nodes	T _i No
✓	3	2
x	6	7

↔

3	6	7	Max # nodes
x	x	x	0
x	x	✓	1
x	✓	x	1
✓	x	x	N/A
✓	x	✓	N/A
✓	✓	x	N/A
✓	✓	✓	N/A

↔

3	6	7	Max # nodes
x	v	v	1
v	x	x	1
v	x	✓	2
v	✓	x	N/A
v	✓	✓	N/A
v	v	x	N/A
v	v	✓	N/A

- Theorem $\# \text{bags} \leq |\mathcal{V}|$ (we can compress tree decomposition if $\# \text{bags} > n$.)
- Theorem Computation time of independent set is $O(2^k n)$.
- Proof Each bag need at most 2^k times.

Fixed Parameter Tractability

Hard problem cannot solve in polynomial time

↓ assure that an input have some properties
(such as treewidth) equal to (small) k

computation time is

$f(k) \cdot n^k$

arbitrary large
function

polynomial of n